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COMPARISON OF HILBERT TRANSFORM AND SINE FIT APPROACHES FOR THE DETERMINATION OF DAMPING PARAMETERS

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Abstract - For the analysis of rotational damping measurements in the time domain, two different identification procedures are compared. The first procedure investigated incorporates a Hilbert transform of the data, which enables an analysis by a linear regression calculation. The second approach is a direct nonlinear regression calculation of a damped sine function. The two approaches are compared using both simulated data and measurement data. The results of the comparison are presented.

Keywords: rotational damping, torsional damping, damping coefficient, decay rate

1. INTRODUCTION

For the dynamic calibration of torque transducers, a model-based approach was developed. To be able to determine the model parameters of the transducer from measurement data, the model properties of the measuring device needed to be determined in advance. Three different kinds of model parameters had to be determined: mass moment of inertia, torsional stiffness, and *rotational damping*. This research was carried out in a joint European research project focusing on the traceable dynamic measurement of mechanical quantities [1].

2. ROTATIONAL DAMPING MEASUREMENT

For the determination of the damping parameters, two methods are available: the analysis of forced vibration in the frequency domain and the analysis of the decay of the magnitude of oscillations excited by an impulse in the time domain. The described methods assume the presence of a single degree of freedom (SDOF) system for the analysis. The damping measurement in the frequency domain is carried out by analysing the half intensity width of the resonance peak of the magnitude transfer function of a system undergoing forced excitations. For the described application to rotational oscillations, the analysis in the frequency domain has some disadvantages compared to the time domain analysis. The generation of a forced rotational excitation in a wide frequency range and with a sufficiently high magnitude level can hardly be achieved. Moreover, parasitic components of radial and axial motion may occur. Additionally, there is a general risk of an overestimation of the damping values for the frequency domain analysis approach [2].

Therefore, the time domain decay analysis was chosen. An impulse excitation generates oscillations with the natural frequency of the analysed system. These oscillations decay with a characteristic rate based on the damping of the system. Assuming viscous damping behaviour, the resulting oscillations can be described as a function of time t with the starting magnitude q_0, the angular natural frequency ω_d of the damped system, the angle offset φ_0, and the decay rate δ following

$$x(t) = x_0 \cdot \mathrm{e}^{-\delta t} \cdot \sin(\omega_\mathrm{d}\, t + \varphi_0) \ . \tag{1}$$

The quantity of interest is the damping coefficient ζ which is related to the decay rate and the undamped natural angular frequency ω_0 as

$$\zeta = \frac{\delta}{\omega_0} \ . \tag{2}$$

The relation between the undamped angular natural frequency ω_0, the damped natural frequency ω_d, and the decay rate is given by

$$\omega_0 = \sqrt{\omega_\mathrm{d}^2 + \delta^2} \ . \tag{3}$$

For a determination of ζ, the analysed methods need to be able to identify both parameters, frequency and decay rate.

It is more complicated to apply a decay analysis of excited oscillations of a mechanical system to rotational oscillations than to translational oscillations. In case of a rotational vibration, the oscillations are generated by a negative step excitation. The negative step emerges from the fracture of a brittle specimen under torsional load. The generated oscillations are measured by means of a non-contact measurement using two rotational vibrometers at the top ($\dot{\varphi}_1$) and at the bottom ($\dot{\varphi}_2$) of the device under test (DUT). More comprehensive information about the measurement set-up and the generation of the oscillations can be found in [3].

3. MEASUREMENT DATA

The two acquired data channels, which contain the angular velocity data from both sides of the DUT, are numerically integrated to be able to derive the time-dependent torsion angle $\Delta\varphi$ of the DUT given by

$$\Delta\varphi(t) = \varphi_1(t) - \varphi_2(t) \ . \tag{4}$$

To eliminate low frequency drifts, the angle difference signal was differentiated afterwards. After low-pass filtering, only a monofrequent sinusoidal signal remains. However, disturbances and noise cannot be avoided completely. The determination of the damping has to be carried out using procedures sufficiently robust for this purpose. Typical measurement data is shown in Fig. 1.

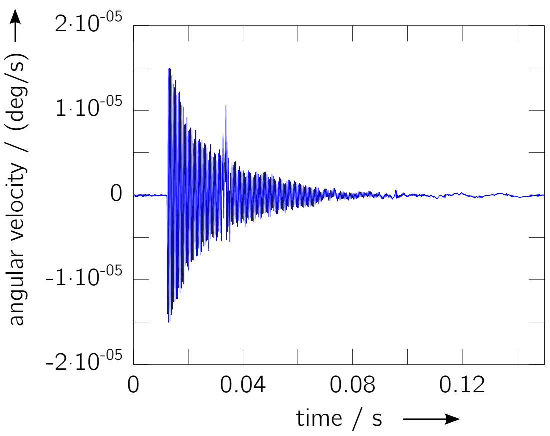

Fig. 1. Rotational damping measurement result.

4. METHODS FOR THE DATA ANALYSIS

The determination of the damping properties from the experimental data can be carried out in different ways. The quantity of interest for the analysis of time series data is not only the decay rate of the signal, but also the oscillation frequency. Only with these two quantities, can the damping coefficient be correctly determined (cf. (2)). The analysis will be carried out with the damped resonance frequency ω_d, influences due to a deviation of the resonance frequency because of the damping will be included in the measurement uncertainty evaluation.

Two methods will be compared regarding their suitability.

4.1. Hilbert Transform

A complex analytic sinusoidal signal $\underline{x}$ contains all information required for the determination of the damping coefficient. It can be described by its real part $x = \Re(\underline{x})$ and its imaginary part $\tilde{x} = \Im(\underline{x})$ following

$$\underline{x}(t) = x(t) + \mathrm{i}\tilde{x}(t) \ . \tag{5}$$

The Hilbert transform $\mathcal{H}$ of the real part of the signal equals a convolution in the time domain [4] giving

$$\mathcal{H}[x(t)] = \tilde{x}(t) = x(t) * \frac{1}{\pi t} \ . \tag{6}$$

In case of a measurement, the acquired signals correspond to the real part of such an analytic signal. The envelope $A(t)$ of the signal can be calculated by means of the acquired signal and its Hilbert transform as

$$\begin{aligned} A(t) = |\underline{x}(t)| &= \sqrt{x^2(t) + \tilde{x}^2(t)} \\ &= \sqrt{x^2(t) + \mathcal{H}^2[x(t)]} \ . \end{aligned} \tag{7}$$

The instantaneous phase $\varphi(t)$ can be derived by means of the same two quantities following

$$\varphi(t) = \arctan\left(\frac{\tilde{x}(t)}{x(t)}\right) = \arctan\left(\frac{\mathcal{H}[x(t)]}{x(t)}\right) \ . \tag{8}$$

The calculus of the arctangent needs to be carried out by a four-quadrant arctangent algorithm (often called *atan2*) to cover the range of the phase result $-\pi < \varphi < \pi$. A subsequent unwrapping of the instantaneous phase removes the phase jump at $-\pi$, π.

The application of the Hilbert transform for the determination of damping data was proposed by [5]. The exponential decay of the envelope

$$A(t) = x_0 \cdot \mathrm{e}^{-\delta t} \ , \tag{9}$$

which has to be estimated (cf. (1), (5)), can be analysed by an equation which is linear in its parameters by applying a logarithmic scale for $A(t)$

$$\begin{aligned} A(t) &= x_0 \cdot \mathrm{e}^{-\delta t} \ , \\ \ln(A(t)) &= \ln(x_0 \cdot \mathrm{e}^{-\delta t}) = \ln x_0 - \delta t \ . \end{aligned} \tag{10}$$

For the natural angular frequency ω_d, a linear relation to the instantaneous phase exists with

$$\varphi(t) = \omega_\mathrm{d} \cdot t \ . \tag{11}$$

Therefore, both quantities required for the determination of the damping constant ζ can be derived by approximating model equations which are linear in their parameters. An application of the method to simulated data is depicted in Fig. 2 and to measurement data in Fig. 3 both for the magnitude and for the instantaneous phase. Depicted in red in both figures is the range of the data which was used for the estimation. With the decay of the signal, the noise becomes dominant and is likely to disturb the estimation if not excluded.

4.2. Sine Fit

The second approach to derive the angular frequency and the decay rate is a direct approximation of (1) with an added bias parameter B giving

$$x(t) = x_0 \cdot \mathrm{e}^{-\delta t} \cdot \sin(\omega_\mathrm{d}\, t + \varphi_0) + B \ . \tag{12}$$

Different from the approach using the Hilbert transform, all quantities can be identified in one regression calculation, which has to be nonlinear due to the sine and exponential terms in the equation.

A direct solution of this nonlinear problem is not possible, thus algorithms suitable for the estimation converge to the solution by iteration. Different from the double linear regression of the method based on the Hilbert transform, the risk of incorrect solutions due to local minima exists.

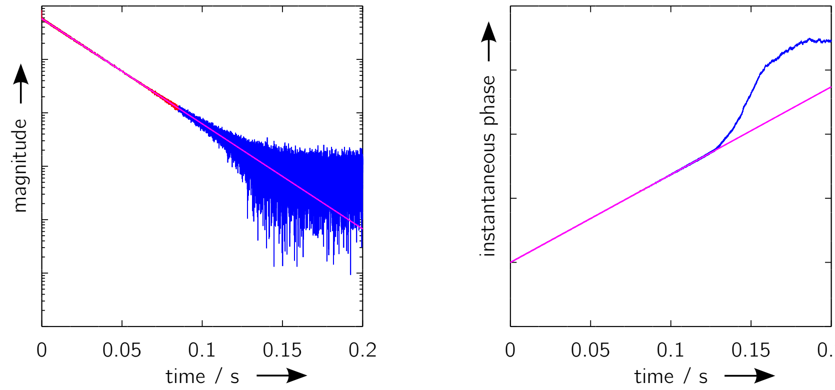

Fig. 2. Measured (blue) and estimated magnitude and instantaneous phase (magenta) of simulated data.

Table 1. Results of the numerical study based on simulated data.

Estimation method	Deviation of the estimated decay rate δ		Deviation of the estimated angular frequency ω_d	
	$\overline{\Delta\delta}_{rel}$	$\sigma_{\delta_{rel}}$	$\overline{\Delta\omega}_{rel}$	$\sigma_{\omega_{rel}}$
Sine fit	$9 \cdot 10^{-9}$	$6 \cdot 10^{-5}$	$-2.4 \cdot 10^{-9}$	$2 \cdot 10^{-7}$
Hilbert transform	$9.9 \cdot 10^{-9}$	$2.3 \cdot 10^{-4}$	$4 \cdot 10^{-7}$	$7.6 \cdot 10^{-7}$

5. NUMERICAL STUDY OF THE PROPOSED METHODS

The suitability of the two approaches was analysed by comparing estimation results from simulated data. For this purpose, a damped sine of well-known properties similar to the measurement results was superposed with random noise. The results of both approaches were compared to the known values used for the data generation.

To investigate the sensitivity to parameter changes, as well as of the quality of the estimation of the decay rate δ, and of the angular frequency ω, repeated simulations with ramdonly altered parameters were carried out. For each of the 10000 data sets used for this comparison, the deviations of the estimated decay rate $\hat{\delta}$ and of the nominal values δ_{nom} were calculated. The same applies to the angular frequencies $\hat{\omega}$, ω_{nom}, accordingly. Based on these outcomes, the mean relative deviations $\overline{\Delta\delta}_{rel}$, $\overline{\Delta\omega}_{rel}$ were calculated following

$$\overline{\Delta\delta}_{rel} = \frac{\overline{\Delta\hat{\delta}}}{\overline{\delta}_{nom}} \quad , \tag{13}$$

$$\overline{\Delta\omega}_{rel} = \frac{\overline{\Delta\hat{\omega}}}{\overline{\omega}_{nom}} \quad , \tag{14}$$

with the corresponding relative standard deviations $\sigma_{\delta_{rel}}$ and $\sigma_{\omega_{rel}}$, respectively. Results for the comparisons are given in Table 1.

It is obvious from the results that both methods show only very small deviations. The numbers are generally a little better for the sine fit but both methods seem to be applicable. A disadvantage of the sine fit are the sporadic irregular results caused by a missing convergence due to local minima. Under the conditions applied in this study, this problem happened in about $0.7\,\%$ of the regression calculations. However, in such a case the results are far from the expected values and do not match the simulated nominal data at all. Therefore, the erratic estimation is easily noticeable and can be excluded from the results.

6. RESULTS OF AN APPLICATION TO MEASUREMENT DATA

Because of the satisfying results of the numerical study, both methods were applied to measurement data. Due to the fact that known values for decay rate and resonance frequency cannot be known, the results of the application of both approaches to the same measurement data file were compared. As a measure for the comparison, the squared residuals of the two fits were chosen. The result of the Hilbert transform does not give all information necessary to reconstruct the measurement signal in a way to compare the residuals to the sine fit results directly (amplitude, bias of the sine signal), therefore, the residuals were calculated based on the envelope of measurement data, of the fitted sine and of the fitted Hilbert transform. An example of a Hilbert fit applied to measurement data is depicted in Fig. 3.

For the comparison, the sum of squared errors (SSE) of each approach was calculated using the residual of each data point. For n data points of measurement data $x_{env,i}$ and the estimation results $\hat{x}_{env,i}$, the SSE was calculated giving

$$\mathrm{SSE} = \sum_{i=1}^{n} \left(\hat{x}_{env,i} - x_{env,i}\right)^2 \quad . \tag{15}$$

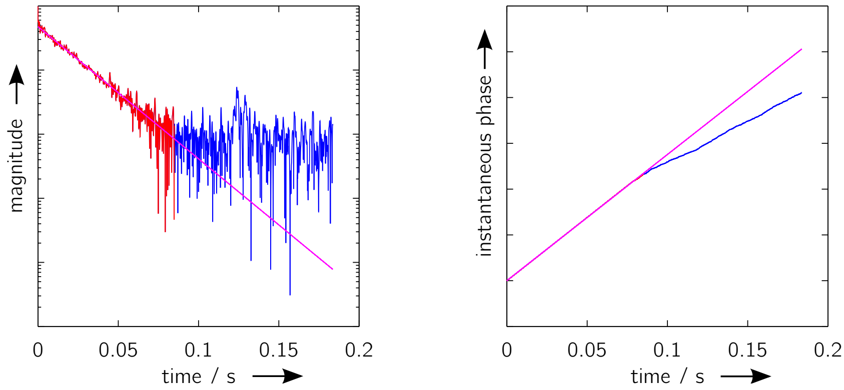

Fig. 3. Measured (blue) and estimated magnitude and instantaneous phase (magenta) of measurement data.

The ratio D of the SSE of the sine fit SSE_{sine} and of the Hilbert fit $\text{SSE}_{\text{Hilbert}}$ was calculated by

$$D = \frac{\text{SSE}_{\text{sine}}}{\text{SSE}_{\text{Hilbert}}}. \tag{16}$$

The results of the comparison of 12 measurement files are given in Table 2. For $D < 1$, the SSE of the sine fit was smaller (i.e. better), for $D > 1$, the SSE of the Hilbert fit.

It becomes obvious from the results of the comparison that both approaches give similar results for most measurement files. However, the Hilbert fit was significantly better only in one measurement, while the sine fit performs better in five cases. The arithmetic mean of the ratio D, $\overline{D} = 0.94$, indicates the same.

Additionally, it has to be kept in mind that the result of a damping estimation based on the method of the Hilbert transform is sensitive to an appropriate choice of the data range. If the noise floor becomes dominant, it will disturb the estimation results. Based on the initial magnitude of each measurement, the data range had to be chosen accordingly to be large enough to have a high number of data points and small enough to avoid influences from the measurement noise.

7. SUMMARY

Two different methods for the estimation of the decay rate and of the angular frequency from measurement data are presented. Both quantities are required to be able to calculate the damping coefficient. The two methods, the first being based on the calculation of an envelope of the oscillations by means of the Hilbert transform, and the second based on a direct nonlinear estimation using a damped sine function, are compared to find out their advantages and disadvantages. This comparison is carried out with simulated data as well as with measurement data.

Because of the slightly better results and because of the missing requirement of a definition of an appropriate data range (which needs to be adjusted for each measurement file), the sine fit approach was chosen for the determination of the damping properties for the dynamic torque calibration device.

Table 2. Results of the comparison based on measurement data.

Measurement	D	Measurement	D
1	0.65	7	0.83
2	1.02	8	1
3	0.93	9	1
4	1.21	10	1
5	0.72	11	1.01
6	0.92	12	0.99

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